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THE ANALYSIS OF AN INTERFEROGRAM

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ABSTRACT. It has been frequently suggested that it should be possible to analyse mathematically the interferometer pattern produced by a lens, in order to obtain a measure of the aberrations from the coefficients of the terms in the various orders of x and y. x and y are here the coordinates of a point on the interferogram, whose optical path difference relative to the central ray of the lens is known at once by counting the fringes. This method has now been tried several times, using the utmost experimental precautions to secure accuracy, but the figures do not agree very closely with those obtained for the same lens under identical conditions by the oblique Hartmann test. Possible reasons for this disagreement are suggested.

THEORY OF THE METHOD

THE process employed in this method of determining oblique aberrations was, briefly, to obtain on a photographic plate an interferogram of the lens, at a certain known obliquity and under suitable conditions of adjustment. The axes of x and y were marked on this plate, by the method to be described, and the distances of the crossing-points of each interference band with each axis were carefully measured. The band nearest the centre was assigned a certain optical-path-difference (O.P.D.), say D; then, since the light travels twice through the lens before producing interference, the O.P.D. of the bands on either side of this will be $D \pm (\lambda/2)$, and of the next bands, $D \pm \lambda$, etc. The sign, + or −, was ascertained by pressing lightly on the instrument behind the convex mirror; this lengthened the path through the lens, making each band move across the lens towards a point of *less* O.P.D. In this way each dark band on the interferogram was marked with its own value of O.P.D., of course in terms of the unknown D which would be subsequently calculated.

The measurements along the two axes gave a series of points whose values of O.P.D. and coordinates $(x_0, 0)$ or $(0, y_0)$ were known. Hence two series of simul-

taneous equations could be constructed, one in x_0 and the other in y_0, by substituting these values in turn into Prof. Conrady's fundamental equation for O.P.D., given in an earlier paper*:

$$\text{O.P.D.} = \frac{a_1}{4}(x^2+y^2)^2 + \left(\frac{h}{l}\right) a_2 y\,(x^2+y^2) + \frac{1}{2}\left(\frac{h}{l}\right)^2 a_3\,(x^2+3y^2) + \frac{1}{2}\frac{\delta l}{l^2}(x^2+y^2) + \frac{\delta h}{l}\,y + \frac{\delta t}{l}\,x \qquad \ldots.(1).$$

For the points lying actually in the x-axis, $y = 0$, and equation (1) reduces to the form

$$\text{O.P.D.} = \left(D \pm n\frac{\lambda}{2}\right) = \frac{a_1}{4}x_0^4 + \left[\frac{1}{2}\left(\frac{h}{l}\right)^2 a_3 + \frac{1}{2}\frac{\delta l}{l^2}\right] x_0^2 + \frac{\delta t}{l}\,x_0 \qquad \ldots.(2)$$

or simply

$$D \pm n\frac{\lambda}{2} = Ax_0^4 + Bx_0^2 + Cx_0 \qquad \ldots.(2\,a).$$

Similarly, for the intersections of the various bands with the y-axis, we have equations of the form

$$\text{O.P.D.} = \left(D \pm n\frac{\lambda}{2}\right) = \frac{a_1}{4}y_0^4 + \left(\frac{h}{l}\right) a_2 y_0^3 + \left[\frac{3}{2}\left(\frac{h}{l}\right)^2 a_3 + \frac{1}{2}\frac{\delta l}{l^2}\right] y_0^2 + \frac{\delta h}{l}\,y_0 \qquad \ldots.(3)$$

or

$$D \pm n\frac{\lambda}{2} = Ey_0^4 + Fy_0^3 + Gy_0^2 + Hy_0 \qquad \ldots.(3\,a).$$

By solving these sets of simultaneous equations, using the method of least squares or the approximate graphical solution outlined below, the most probable values of A, B, C, etc. could be determined, and hence values obtained for a_1, a_2, etc. Since, however, only a_1, a_2, and a_3 are ultimately required, the other unknowns δh, δl, δt, and D could have been directly eliminated. The least square method, moreover, gives a measure of the weights of the determined quantities, from which their probable errors can be determined.

The Experimental Apparatus

The actual interferometer itself needs no further description†, but one or two accessories were fitted for these experiments. Firstly, monochromatic light of the greatest possible brilliancy was required to reduce the exposure to a reasonable time, and this was also required to be the mercury violet light (λ = ·0000436 cm.) in order that the results of this test should be comparable with those of the oblique Hartmann test on the same lens, described to this Society recently‡. At first a violet filter was used with a mercury vapour lamp, but this absorbed so much light that it was decided to erect a simple monochromator to throw a mercury spectrum on to the interferometer pinhole plate, so arranged that the violet line fell on the

* R. Kingslake, *Trans. Opt. Soc.* **27** 1925–6 (95).
† F. Twyman, *Phil. Mag.* **35** 1918 (49).
‡ R. Kingslake, *Trans. Opt. Soc.* **27** 1925–6 (228).

actual pinhole itself (Fig. 1). In this way the necessary exposure, using an Ilford "Monarch" plate (H. and D. 525) could be reduced to as little as two minutes.

The apparatus was set up in a dark room, and at first the photographic plate was placed in a lensless camera erected beyond the eye-hole; but a few preliminary experiments showed the presence of a very large error, which was finally tracked down to the spherical aberration in the unachromatised collimator and viewing-lenses. This acts as follows:

(*a*) In the viewing-lens, we have parallel light carrying the interference pattern falling on the lens and coming to a focus, beyond which the widening beam is intercepted by the plate. Evidently a perfect lens would give a true-to-scale rendering of the pattern in this way, but spherical aberration has the effect of excessively magnifying the outer parts of the picture.

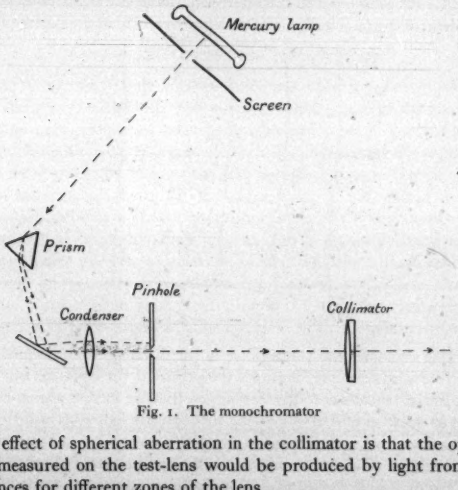

Fig. 1. The monochromator

(*b*) The effect of spherical aberration in the collimator is that the optical path differences measured on the test-lens would be produced by light from different object-distances for different zones of the lens.

It was therefore decided to replace the collimator with an achromatic telescope objective of the same focal length, and to remove the viewing-lens from the system by fixing the plate to the front of the viewing-lens cell with a spot of wax. To secure good collimation, the "pinhole" was reduced from 1 mm. diameter as supplied on the instrument, by sticking over it tinfoil pierced with a fine needle-point. The focussing of the collimator was checked by taking two photographic sections of the beam at two distances from the collimator lens. On placing these plates face to face it was seen at once that the beam was neither expanding nor contracting.

Correction for Change of Wave-shape

This trouble is much more subtle, and is inherent in the nature of the light-waves themselves. It appears as a gradual change in the form of the interference pattern as it progresses from the test-lens towards the observer, so that a photograph taken on a plate inserted in the emerging beam close to the lens will differ materially from one taken farther away. The reason for this change is that by Huygens' principle a wave-front is always propagated parallel to itself, along the normal at each point. A truly spherical wave will remain spherical, but a wave distorted by aberration will change its shape as it moves, convex parts becoming larger and concave parts smaller (see Fig. 2). In the present work, therefore, the plates for measurement were taken about 25 cm. from the lens, and a correction was calculated for each band to transform the whole pattern back to what it would be in the plane of the lens itself. The distance over which the pattern had travelled

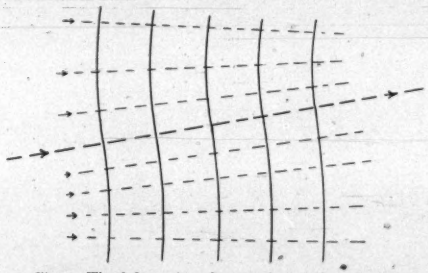

Fig. 2. The deformation of a wave-front as it progresses

while changing its shape was taken as being the distance from the plate back to the convex mirror; this, however, includes the whole of the distance from the lens to the convex mirror and back to the lens again, because in that region the distortion is only half as great as it is after passing through the lens a second time. For this same reason, also, a straight line ruled across the lens will in general appear curved on the photograph. The actual magnitude of this disturbing effect depends on the number of bands visible, being almost negligible in presence of four or five bands, but becoming decidedly large if the pattern contains, say, twenty bands.

As hardly any lens appears perfect over its whole aperture when examined in the interferometer, it was desirable to ascertain the maximum aperture of the lens under test over which it would be safe to select points for the determination of the oblique aberrations. When the lens was exactly square-on, the interferogram (Fig. 3) consisted of one ring very near the margin of the lens, but if it was now very slightly tilted, the series of bands were quite straight across the centre of the pattern and then suddenly bent round at the ends (Fig. 4). The diameter of the good part of the lens containing the straight bands was 2·5 cm., so it was decided

to accept values of x_0 and y_0 only up to about 12·5 mm. from the centre. In this way the doubtful marginal part of the lens would be eliminated.

Fig. 3. The "best" interference pattern

Fig. 4. The useful aperture of the lens

Locating the Axes of x *and* y *on the Plate*

The success of the whole method depends very largely on the accurate marking of these axes across the photographic plate carrying the interferogram. Since the pattern always appears bounded by the lens aperture itself, it would seem at first sight sufficient to stretch a pair of very thin wires across the aperture so that a shadow of these would be seen on the photograph; one wire coinciding with the axis of tilt of the lens and the other perpendicular to it. But attempts along these lines were unsuccessful owing to the great broadening of the shadows by diffraction effects. Eventually the complete wires were used in all the preliminary adjustments, and the central parts were then cut away, so that only the diffuse shadows of the remaining ends of the wires appeared on the photograph. These shadows were joined on the plate by fine scratches to mark the axes for measurement. The positive direction of y was marked in the direction of the height of the image above the axis, and the positive direction of x would then bear the usual geometric relation to the y-direction, as viewed from the image side of the lens. Owing to the reflection in the half-silvered mirror, the usual right-handed relation of the positive x and y directions would also be seen by looking into the instrument in the ordinary way or through the glass side of the photographic plate.

As was mentioned above, the tilt of the lens necessarily made the x-axis appear curved on the plate, while the y-axis appeared straight but the points on it would need correction. To determine the curvature of the x-axis experimentally, the lens was accurately squared-on by securing the pattern with the minimum number of bands, and the plane mirror of the interferometer was then covered up. A plate was now taken, which of course showed both cross-wires as straight shadows. The lens was then tilted to its final amount of $5° 3'$ and another photograph taken, which showed the y-axis straight and the x-axis curved. The plates were placed in superposition and the central separation of the x-axes measured, from which the radius of curvature was determined by the spherometer formula. A beam-compass was set to this radius (56 cm.), and a curved scratch was made on the plate

for the x-axis, joining the shadows of the ends of the cut-away cross-wire, a straight scratch being made similarly for the y-axis.

The values of x_0 and y_0 for all the bands were then determined by setting on the crossing-points of the successive bands and the axes with a measuring microscope (Fig. 5). The marks + and − show the sign of the O.P.D.

To obtain the corrections to the values of y_0, a large-scale graph was drawn connecting O.P.D. with the observed values of y_0, keeping the O.P.D. scale as small as possible in order to minimise the effects of the disproportionality between the scales of wave-lengths for O.P.D. and mm. for y_0. The slopes of the curve at the positions of the various bands were now read off this graph, the inclination being reckoned relative to the tangent at the origin; for, since the apparatus was purposely

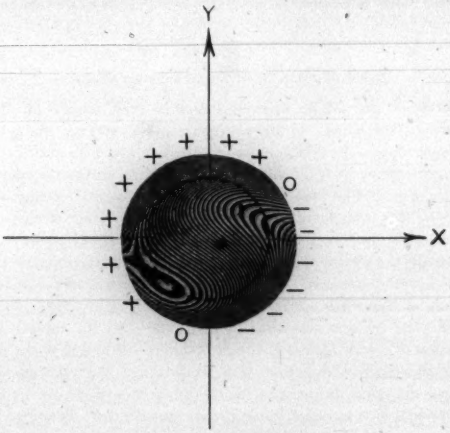

Fig. 5. The final plate

adjusted to give a clear single shadow of the centre of the cross-wires, it is evident that the light passing through the origin of the lens is travelling parallel to the optical axis, and therefore this point and this point only can be regarded as definitely fixed, no matter how far away the plate is situated.

The actual linear corrections to y_0, δy_0 were calculated by multiplying the values of the slope at each point by 28 cm., which was the distance from the plate to the surface of the convex mirror, and these corrections were added algebraically to the values of y_0 before performing the main calculation. The reason why the corrections must be *added* is clear from Fig. 6, in which the slope is shown positive, requiring addition of the correction δy_0 to the observed y_0 to give the corresponding value of y_0 at the lens itself.

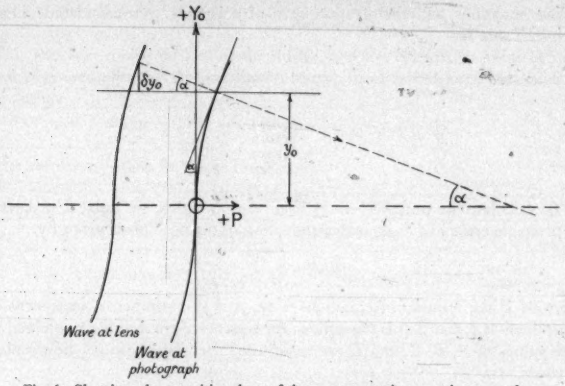

Fig. 6. Showing why a positive slope of the curve necessitates an increase of y_0

Analysis of the Measurements

(*a*) *The* x*-axis Intersections*

As we have seen, no correction was necessary in this case, the calculations being made with the figures obtained directly with the measuring microscope. These are given in Table I.

Table I

O.P.D.	x_0	O.P.D.	x_0
λ	cm.	λ	cm.
$D - 4{\cdot}5$	·9768	$D + 1$	− ·2058
$D - 4$	·8671	$D + 1{\cdot}5$	− ·3038
$D - 3{\cdot}5$	·7564	$D + 2$	− ·4038
$D - 3$	·6439	$D + 2{\cdot}5$	− ·5075
$D - 2{\cdot}5$	·5340	$D + 3$	− ·6027
$D - 2$	·4253	$D + 3{\cdot}5$	− ·6968
$D - 1{\cdot}5$	·3146	$D + 4$	− ·7892
$D - 1$	·2117	$D + 4{\cdot}5$	− ·8871
$D - {\cdot}5$	·1064	$D + 5$	− ·9777
D	·0046	$D + 5{\cdot}5$	− 1·0692
$D + {\cdot}5$	− ·1051	$D + 6$	− 1·1626

As has been shown, these figures satisfy an equation of the form

$$D \pm n\frac{\lambda}{2} = Ax_0^4 + Bx_0^2 + Cx_0 \qquad \ldots(2a).$$

Our twenty-two values of x_0 give twenty-two equations for A, B, and C. By the method of least squares*, the most probable values of A, B, and C satisfying

* Chauvenet, *Spherical and Practical Astronomy*, vol. **2**, appendix.

all these equations, with their respective relative weights, were calculated. They are given in Table II.

Table II

Coefficient	Value (λ)	Weight
A	·021464	·3486
B	·243016	·4062
C	− 4·842257	7·5375
D	− ·004829	6·4937

To calculate the probable errors from these weights, we must first determine the probable error ϵ of a single determination of O.P.D. This is given by

$$\epsilon^2 = \frac{\Sigma v^2}{M - \mu},$$

where M is the number of equations = 22, μ is the number of unknowns to be determined = 4, and Σv^2 is the sum of the squares of the residuals obtained when these values of A, B, C, and D are substituted successively in all the twenty-two equations.

The value of ϵ was found to be ·0130λ. This shows the order of accuracy with which the *observed* points fit the *mean* curve, and it incidentally gives us an idea as to the required degree of accuracy in the mirrors of the interferometer. If the experimental errors are only about $\frac{1}{80}\lambda$, it appears that the accuracy of about $\frac{1}{10}\lambda$, which is generally considered sufficient for the mirrors, is not nearly good enough for the measurement of aberrations with this precision.

However, assuming a perfect instrument, we will determine the probable experimental errors of A, B, C, and D due to errors in measurement and errors in marking the axes of x and y on the plate, etc.

The probable error of A is given by $\frac{\epsilon}{\sqrt{\text{weight of } A}}$, so we have finally the values shown in Table III.

Table III

Coefficient	Value	Probable error	Value	Probable error
	λ	λ	cm.	cm.
A	·021464	·02202	·0000009354	·000000960
B	·243016	·02040	·0000105906	·000000889
C	− 4·842257	·00474	− ·0002110256	·000000207
D	− ·004829	·00510	(no interest)	

The full line graph in Fig. 7 shows the result of plotting O.P.D. against x_0, and is the section of the wave-front by the plane $y = 0$. The slight curvature is made up of a small square term together with a much smaller fourth power term. The slope at the origin is negative, confirming the tilt given to the mirror-carriage before taking the photograph. Since the probable error of D is greater than the value determined for D, it is evident that the location of the origin as the intersection of the two scratched lines is not exactly correct.

(*b*) *The y-axis Intersections*

This axis was measured just as the x-axis, and the corrections were obtained from a large-scale graph. As was shown above, these points satisfy an equation of the form

$$D \pm n\frac{\lambda}{2} = Ey_0^4 + Fy_0^3 + Gy_0^2 + Hy_0 \qquad \ldots.(3\,a).$$

The results are given in Table IV.

Table IV

O.P.D.	Observed y_0	Correction δy_0	Corrected y_0
λ	cm.	cm.	cm.
$D + \cdot5$	1·0885	·0219	1·1104
D	1·0455	·0199	1·0654
$D - \cdot5$	·9933	·0198	1·0131
$D - 1$	·9330	·0180	·9510
$D - 1{\cdot}5$	·8554	·0158	·8712
$D - 2$	·7439	·0141	·7580
$D - 2$	·4098	·0044	·4142
$D - 1{\cdot}5$	·2710	·0037	·2747
$D - 1$	·1666	·0019	·1685
$D - \cdot5$	·0800	·0006	·0806
D	·0025	·0000	·0025
$D + \cdot5$	− ·0729	− ·0004	− ·0733
$D + 1$	− ·1485	− ·0008	− ·1493
$D + 1{\cdot}5$	− ·2224	− ·0024	− ·2248
$D + 2$	− ·2956	− ·0010	− ·2966
$D + 2{\cdot}5$	− ·3639	·0010	− ·3629
$D + 3$	− ·4397	·0015	− ·4382
$D + 3{\cdot}5$	− ·5152	·0024	− ·5128
$D + 4$	− ·5936	·0027	− ·5909
$D + 4{\cdot}5$	− ·6849	·0032	− ·6817
$D + 5$	− ·7820	·0044	− ·7776
$D + 5{\cdot}5$	− ·9139	·0053	− ·9086

The method of least squares was applied to these figures, giving the following values and probable errors for the coefficients (Table V). In this case the value of ϵ came to be ·0128λ, which is practically the same as for the x-axis.

Table V

Coefficient	Value	Weight	Probable error	Value	Probable error
	λ		λ	cm.	cm.
E	·100486	·2142	·02774	·0000043792	·000001209
F	3·204804	·7997	·01436	·0001396654	·000000626
G	2·472556	·2037	·02844	·0001077540	·000001239
H	− 6·386428	·7101	·01524	− ·0002783205	·000000664
D	·001978	2·1419	·00877	(no interest)	

This value of D does not agree very well with that obtained by the x-axis, but since the probable error has come out about four times as great as the quantity itself, it simply shows that the experimental location of the origin is very uncertain.

The value of D is, however, extremely small; if it had happened to be larger, i.e. if the nearest band to the centre had been some distance from it instead of practically crossing over it, the proportional error would have been much less although the actual error would have been about the same.

The broken line graph in Fig. 7 is the curve obtained by plotting O.P.D. against the corrected y_0. This is a section of the wave-front as it leaves the lens by the plane $x = 0$. The shape of the curve is mostly due to the large cubic term due to the coma present. The slope at the origin is negative, again confirming the adjustment given to the mirror carriage.

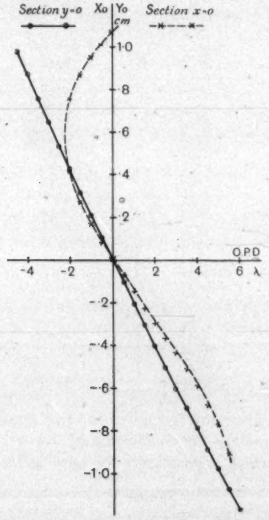

Fig. 7. Cross-sections of the wave-front

Evaluation of the Aberration Constants

Before it was possible to calculate the values of a_1, a_2, etc. from these results, it was necessary to measure the image-height h and the focal length f. The latter was found by means of the focal-collimator to be 30·55 cm. To determine h, the angular tilt of the objective on its geometric tilting-device was measured by erecting the whole arrangement in a vertical plane on the table of a spectrometer and sticking a small mirror to the lens with wax. At each extremity of the tilt employed, a reflection was obtained in this mirror from an auto-collimating eye-

piece mounted in place of the telescope eyepiece. The value of θ so obtained ($5^\circ\ 3'$) gave

$$h = f \tan\theta = 2{\cdot}700 \text{ cm.}$$

Now by equations (2) and (3), we have

$$\left.\begin{aligned} A &= E = \frac{a_1}{4} \\ B &= \frac{1}{2}\left(\frac{h}{f}\right)^2 a_3 + \frac{1}{2f^2}\delta l \\ G &= \frac{3}{2}\left(\frac{h}{f}\right)^2 a_3 + \frac{1}{2f^2}\delta l \end{aligned}\right| \text{ whence } \left\{\begin{aligned} G - B &= \left(\frac{h}{f}\right)^2 a_3 \\ 3B - G &= \left(\frac{1}{f}\right)^2 \delta l \end{aligned}\right.$$

$$C = \frac{1}{f}\delta t$$

$$F = \frac{h}{f} a_2$$

$$H = \frac{1}{f}\delta h$$

We will consider the various coefficients separately.

(*a*) *Spherical Aberration.* This is given by the coefficient a_1. Now

$$a_1 = 4A = {\cdot}0000037 \pm {\cdot}0000038$$

and

$$a_1 = 4E = {\cdot}0000175 \pm {\cdot}0000048$$

which shows at once a great uncertainty in the value of a_1. This is due entirely to the small amount of aberration present, for the probable error would be less in absolute magnitude if the amount of aberration were greater. In any case, this entire range of uncertainty is only about a quarter of the Rayleigh tolerance for a_1 of ·000061. It is difficult to see why the value of a_1 determined from the y-axis should be so much greater than that from the x-axis, but there may possibly be a trace of secondary oblique aberration present which would not appear on the x-axis at all, but would be included in the fourth power term of the y-axis. This method really gives very little information about so little spherical aberration as this lens possesses.

(*b*) *Coma.* Coma is given by

$$a_2 = F \Big/ \left(\frac{h}{f}\right) = {\cdot}001580 \pm {\cdot}000007.$$

This is a range of uncertainty of ·44 per cent., of course excluding any errors in the interferometer mirrors themselves.

(*c*) *Astigmatism.* For this

$$a_3 = (G - B) \Big/ \left(\frac{h}{f}\right)^2 = {\cdot}01244 \pm {\cdot}00027$$

which is a range of uncertainty of 2·2 per cent.

(*d*) *The displacements*. These come out as follows:

$$\delta l = f^2(3B - G) = -\cdot0709 \pm \cdot0036 \text{ (5 per cent.)}$$
$$\delta t = f.C = -\cdot0064 \pm \cdot000006 \text{ (0·1 per cent.)}$$
$$\delta h = f.H = -\cdot0085 \pm \ 00002 \text{ (0·2 per cent.)}$$

Conclusions

Provided that the actual amounts of aberration present are reasonably large compared with the tolerance, that sufficient care is taken in locating axes, etc., as outlined above, and that it is safe to assume only primary aberrations are present, it should be possible to measure aberrations to about 1 or 2 per cent., assuming a perfect instrument.

In order that instrumental errors shall not exceed this experimental error, the mirrors must nowhere introduce errors of more than ·01 wave-length. This is, unfortunately, hardly likely to be attained in practice, as surfaces, silvering, homogeneity of the half-silvered plate, movements of the air, elastic creep of the mountings, etc., may all produce much larger errors than this in ordinary use. It would give more faith in the results if there were some means provided on the interferometer itself for testing the accuracy of all the mirrors involved.

Comparison of Interferometer and Hartmann Tests

The figures obtained for the same lens by the two methods form an interesting comparison (Table VI).

Table VI

Coefficient	Interferometer test	Hartmann test
a_1	(very uncertain)	(not determined)
a_2	·001580 ± ·000007 (·44 per cent.)	·001752 ± ·000008 (·45 per cent.)
a_3	·01244 ± ·00027 (2·2 per cent.)	·01598 ± ·00019 (1·1 per cent.)

As regards astigmatism, the theoretical value of a_3 for a thin lens is $1/2f = \cdot0164$; but we should expect the values determined by these experiments to be slightly less, as the diaphragm or cross-wire was rather nearer the image than the position of the equivalent thin lens. But it is very strange that the interferometer should give values of both coma and astigmatism so much smaller than the Hartmann test. It should be again pointed out that the probable errors here given for the interferometer result are experimental errors only, and take no account whatever of any errors in the mirrors of the instrument itself. It need not be surprising therefore if the actual errors were up to ten times those given in this table; but even this would not bring the values up to those found by the Hartmann test.

Further experiments were made on rotating the convex mirror, and it was found that in certain positions marked axial astigmatism appeared, which of course would be included in the analysis as lens aberration of the oblique pencil. If there were, by chance, errors in the mirror roughly proportional to the cube of the height, this would appear as coma in the analysis.

The interferometer results are very difficult to obtain and rest on so many slender foundations and grave assumptions. The Hartmann test, on the other hand, is simple and straightforward in the experimental work, and still more so in the calculation of the aberration constants. If the interferometer mirrors could be made accurate to ·01λ the two methods would have comparable accuracy. The chief use of the interferometer method is the detection and figuring of irregularities in surfaces and glasses, and in the visual detection of aberrations in a qualitative, or roughly quantitative, way.

Approximate Graphical Method of determining the Aberration Coefficients

The amount of work involved in the least-square method as described above is really very considerable, and it was thought desirable to include a description of a rapid graphical method of obtaining this information in a few minutes, to an accuracy of about 8 or 10 per cent.

The method is based on the fact that the derivative of a linear expression is also linear, but the degree of each term is one less than it was before.

Consider equation (3 *a*); this reads

$$\text{O.P.D.} = Ey_0^4 + Fy_0^3 + Gy_0^2 + Hy_0 \qquad \ldots.(\text{i}).$$

If we write O.P.D. as P and y_0 as Y, and then plot O.P.D. against y_0, we shall get a curve having the equation

$$P = EY^4 + FY^3 + GY^2 + HY \qquad \ldots.(\text{ii}).$$

This has actually already been done (see Fig. 8).

Suppose that by some means we now plot the derived curve of this, i.e. the curve whose ordinates are everywhere proportional to the *slope* of the original curve at that point. Then the equation of this slope-curve, or derived curve, will be

$$P' = \frac{\partial P}{\partial Y} = 4EY^3 + 3FY^2 + 2GY + H \qquad \ldots.(\text{iii}).$$

Then it is at once evident that H will be the value of the ordinate of this derived curve at the point $Y = 0$; or the length of the intercept of this curve along the P-axis.

If we now draw the "second derived curve," or the slope-curve of this first derived curve, it will have the equation

$$P'' = \frac{\partial^2 P}{\partial Y^2} = 12EY^2 + 6FY + 2G \qquad \ldots.(\text{iv}),$$

and the ordinate of this curve at the point $Y = 0$ will have the value $2G$. Similarly the ordinates of the third and fourth derived curves at $Y = 0$ will have the values $6F$ and $24E$ respectively. Hence by drawing all these four derived curves, the values of E, F, G, and H can be very quickly determined.

The details of the process are best explained by describing an actual example, the case chosen being that of the Y-axis considered already. The corrected values

of y_0 were plotted (Fig. 8) to a scale of $\frac{1}{2}$ inch to one wave-length of O.P.D. in the usual vertical or y-direction; and 5 in. for one cm. of y_0 in the horizontal direction. As smooth a curve as possible was drawn through these points; this is called the "curve of P." To derive from this the curve of $\partial P/\partial Y$, a polar distance QM was drawn along the prolongation of the horizontal axis, and a vertical line ef was drawn at M. A line was now drawn through Q parallel to a short section ab of the given curve, to meet ef at c. Then the height Mc is proportional to the slope of ab, being in fact the actual slope of ab multiplied by the polar distance QM. A height Mc was therefore marked off along the mid-ordinate of the short length ab, thus giving the point d on the derived curve. By repeating this process for all such little elements of the curve of P, a series of points d could be found, which when joined by a smooth curve, gave the desired "curve of $\partial P/\partial Y$."

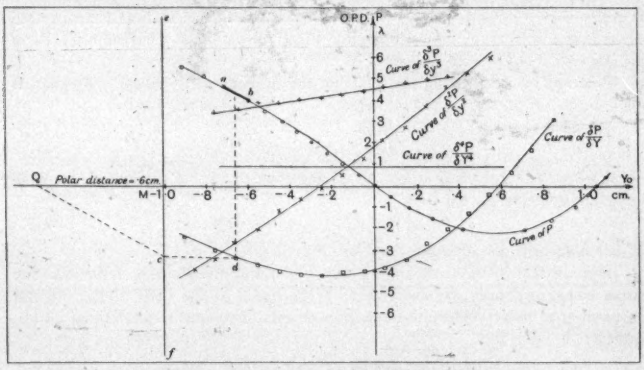

Fig. 8. Graphical solution of aberration coefficients for y-axis

As regards choice of polar distance, this should be large in order to give an easily measured intercept on the vertical axis; but if it is too large the curve is liable to run off the paper. It is best to find the steepest part of the given curve first, and to choose the polar distance so that the ordinate of this part of the derived curve will be a convenient size. The subsequent curves may either be drawn with the same polar distance or with another more convenient one.

In Fig. 8 the second, third, and fourth derived curves were drawn in the same way, all to the same polar distance of 3 in. actual, representing to scale a distance of 0·6 cm. of y_0. It should be noted that the true value of $\partial P/\partial Y$ at any point on the curve of P is given by the ordinate of the curve of $\partial P/\partial Y$ at that point *divided* by the polar distance. Similarly the slope of the $\partial P/\partial Y$ curve is given by the corresponding ordinate of the curve of $\partial^2 P/\partial Y^2$ divided by the *square* of the polar distance since it has now been applied twice, and so on.

The measurements from the graphs gave these results:

Curve	Intercept on vertical axis	Proportionality constant	True value
	λ		λ
P	0	1	0
$\partial P/\partial Y$	− 3·89	$1/\cdot 6 = 1\cdot 667$	$H = -6\cdot 49$
$\partial^2 P/\partial Y^2$	1·64	$1/(\cdot 6)^2 = 2\cdot 778$	$2G = 4\cdot 57$
$\partial^3 P/\partial Y^3$	4·50	$1/(\cdot 6)^3 = 4\cdot 630$	$6F = 20\cdot 89$
$\partial^4 P/\partial Y^4$	0·86	$1/(\cdot 6)^4 = 7\cdot 714$	$24E = 6\cdot 64$

These figures may now be compared with those obtained by calculation.

Coefficient	Value by graph	Value by calculation	Difference
	λ	λ	λ
D	0	·002	− ·002
H	− 6·49	− 6·386	− ·104
G	2·28	2·472	− ·192
F	3·48	3·205	·275
E	·28	·100	·180

Evidently the error of this method is of the same order, no matter what is the size of the quantity being measured; percentage errors would therefore be very misleading, but in an average case, with a fair amount of aberration present, the result would be within about 8 per cent.

One other property of this method is of interest, namely, that it will detect the presence of higher aberrations. In the case chosen as an example the highest term present in the original equation is the fourth, so that the curve of $\partial^3 P/\partial Y^3$ is an inclined straight line, and the curve of $\partial^4 P/\partial Y^4$ is a horizontal straight line. Had higher terms been present, corresponding to the presence of higher aberrations in the lens, then the graph of $\partial^3 P/\partial Y^3$ would have been slightly curved, and the $\partial^4 P/\partial Y^4$ line would have been inclined, or in an extreme case, even this might have been curved too.

DISCUSSION

Mr T. Smith: Mr Kingslake has given us a very interesting account of his attempt to derive the aberrational coefficients of a lens from a single interferogram. His paper will be of much interest to a large number of members of this Society. It raises, perhaps to a greater extent than Mr Kingslake realised when he prepared it, a number of controversial questions. I can only touch on one or two of these here, and say that on some points I agree, and in others disagree with the conclusion of the paper.

It should be realised that in carrying out this work Mr Kingslake had not the advantage of using a Hilger camera lens interferometer. That added greatly to the difficulties he had to contend with on the one hand, and on the other indicates that some of the remarks he makes are not necessarily applicable to the instrument specially made for this work. We know in fact that the camera lens interferometer

has proved a decided success as a tool for the precise evaluation of aberrations. In justice to Messrs Adam Hilger, Ltd., it must be emphasised that Mr Kingslake has made demands upon a workman's tool which it was never intended to meet: and the excellence of the results Mr Kingslake has obtained—for in my opinion they fully merit that term—is attributable very largely to the care he has taken and to his perspicacity in discovering the origin of errors which arise when this workman's tool is employed in this way. I very much hope that the time is not long distant when the Technical Optics Department of the Imperial College will be equipped with a Hilger camera lens interferometer; when that has been installed I am sure that many investigations will be carried out with its aid in which this Society will be specially interested.

I must here restrict myself to a few points arising out of Mr Kingslake's paper. The method of least squares and the theory of errors have fields in which they may quite legitimately be employed, but, speaking very roughly, they are not—at least in the way commonly seen—applicable in physical measurements. The so-called probable error is not infrequently of an altogether different order of magnitude from the probable accuracy of the result achieved. It must be remembered that the theory assumes that all systematic errors are known and removed, and that the unsystematic errors arising from any one case are infinitesimal in magnitude. I should unhesitatingly reject Mr Kingslake's conclusion that the spherical mirror should be surfaced correctly to $\frac{1}{80}$ of a wave-length. An accuracy of roughly $\frac{1}{10}$ of a wave-length—certainly $\frac{1}{20}$—is sufficient for all work of this character.

From equations (2 *a*) and (3 *a*) the quantities $A, B, C, D, E, F, G, H, P$ are of various dimensions ranging from (length)$^{-3}$ for A and E to simple length for D and P. Tables II, III, and V appear to suggest that they may all be regarded as lengths. Is it the case that the headings in these tables are intended to mean that the numerical values quoted are to be given to these quantities to yield the path differences in wave-lengths or centimetres as the case may be when x and y are in centimetres?

Perhaps Mr Kingslake's value (·000061) for the Rayleigh tolerance for a_1 given on p. 11 is due to a_1 not being a length. For the aperture of 2·5 cm. taken in these observations I find the smaller value ·000018. This of course disregards the possibility of correction by change of focus, which is quite a distinct question.

Arising from this point I should like to point out that a system of aberrational coefficients in which each coefficient is of different dimensions is extremely inconvenient both practically and theoretically. The only systems which appear generally useful are that in which all the coefficients are lengths, say wave-lengths, and that in which they are pure numbers. Of the two the latter is in my opinion the better. In the notation of the paper if it is known that the A of one system is a thousand times as great as the A of another it is impossible to say which of the two, as regards the removal of spherical aberration, is better designed or corrected. Other particulars about the systems and the measurements have to be given, and calculations made, before a comparison can be made. Any corrections which have to be made for object and stop position errors are also more troublesome.

The last point to which I wish to refer is Mr Kingslake's comparison of the results yielded by the interferometer test and the Hartmann test. If I understand Mr Kingslake's procedure aright, his comparison is not quite legitimate. In the Hartmann test the co-ordinates x and y refer to points in the plane of the perforated disc, which is situated in the image space. In the interference test, since the lens is viewed from the object side, the reference plane is in the object space. Moreover, both are real—in the present investigation the plane is determined by wires—and they cannot therefore be simply in an object and image relation. If my supposition is correct, the aberrational coefficients of the two investigations, though denoted by the same letter, are essentially different quantities, and the paper does not conclusively establish any difference in the aberrations disclosed by the two methods. My inference from Mr Kingslake's work would then be that there is a very distinct hope that the analysis of an interferogram will give measures of aberrations of an accuracy as high as the physical character of light permits us to regard as significant.

I infer from the paper that Mr Kingslake is inclined to the view that the Hartmann test is likely to be more accurate than an interferometer test. I am disposed to adopt the opposite view, and in case of difference to attach greater weight to the interference pattern. A discussion on this point would be out of place here, and I hope to be able to speak with greater confidence on this point—if necessary to revise my opinion—after we have made a considerable number of investigations with the camera lens interferometer recently installed at the National Physical Laboratory.

Mr J. W. Perry (communicated): The method adopted by Mr Kingslake differs somewhat from methods of measurement by reference to the focal plane, and involves disturbing circumstances which have been well surmounted by Mr Kingslake. It may be remarked that in like investigations at the laboratories of Adam Hilger, Ltd., the first stage of the procedure has been to trace fine ink-lines on chosen suitable surfaces of the lens, the latter being mounted in the lathe for that purpose. In this way, since the position of such zones may be measured very accurately and independently of the test, the corrections discussed by Mr Kingslake in this connection may be avoided.

Beyond the actual method the attention of the author appears to centre largely around relative and comparative errors. It is then almost difficult not to mistake Mr Kingslake where the relation of the Rayleigh limit to his own as here put forward is involved: and inasmuch as Mr Kingslake does not admit to the establishment of his limit any reference to the physical significance thereof, a serious potential detraction from its value lies in the recognition that actual aberrations may in most cases exceed that limit more than 100 times, viz., may exceed 1λ, before the Rayleigh standard of appreciable deterioration of the image is approached. The errors calculated have, of course, a precise meaning in relation to the total conditions obtaining when the measurements were made. It must, though, not be supposed that such values possess general significance. It is indeed very probable that in course of time Mr Kingslake would acquire experience and increased

facility in applying the method which he puts forward, such as would lead to his further refining and perfecting it—for example by distributing the points of measurement in the most suitable manner over the interferogram and by continued attention to external conditions: so that one may anticipate that these limits would eventually call for revision, even until almost the inevitable irreducible minima were approached, corresponding to the symmetrical non-systematic irregularities in the permanent physical conditions of the test, including as part thereof the state of the instrument used and the instrumental errors, both inherent and accidental.

It is clear therefore that Mr Kingslake's paper furnishes the following noteworthy result, viz., that the mean final non-uniform or discontinuous irregularities resulting from remaining heterogeneities in present-day optical glass and from the traits of craftsmanship now do not exceed $\frac{1}{80}\lambda$ in an optical instrument of the type in question. More than this it is not possible to conclude with certainty from these error determinations.

There further remain, here as in the Hartmann or any other test, the systematic errors. Such errors can never be completely eliminated, but the systematic errors of the instrument could be accounted for by simply imparting to any mirror a small known displacement; from the result of which upon the interferogram the corresponding contribution to the total systematic error may be determined. But it is not thought that in the course of practical testing the determination would be, or would need to be, extended and forced to such refinement as is here indicated, for obvious reasons of economy and consistency.

In regard to the discrepancy between the determinations made by the two methods, it is clear that in view of the magnitude of the discrepancy the most probable explanation lies in an accumulation of small systematic errors. Such unfortunate circumstances are of course unavoidable: and in any given case, advantage in eliminating such errors by corrections would clearly rest with the method implicating to the least extent obscure and but partially formulated physical phenomena such as the relative displacement of photographed images known to be due to the distortion effect in gelatine films.

Nothing occurring in this paper can therefore be understood as valid criticism against the practical applicability and general methods of use of either test: and the methods given and results obtained, because of their intrinsic value, will be much appreciated by its users as a further application of the interferometer.

Mr Kingslake: I wish to thank both Mr Smith and Mr Perry very much for their helpful criticism and remarks on this paper. Undoubtedly the interferometer at present in the Optics Department of the Imperial College was originally intended as a workshop tool, but the present work was undertaken partly with the idea of discovering to what extent a simple *non-measuring* interferometer, such as that possessed by this Department, could be used for quantitative work, without the necessity of adding any micrometer screws or other appliances for the direct measurement of the image itself. As regards the necessary accuracy of the mirrors,

for camera lenses where the aberrations expressed as differences of optical path may reach several whole wave-lengths without upsetting the definition, obviously a mirror-error of $\lambda/10$ will be inappreciable; but in the present case I was attempting the far more delicate problem of measuring the aberrations of a telescope objective, where the Rayleigh limit of $\lambda/4$ is a very real thing. A local mirror-error of $\lambda/10$ will produce a change in the optical path of a ray reflected from that part of the mirror, of twice this amount, or $\lambda/5$, which is already almost the entire Rayleigh limit. My demand for mirrors having an accuracy of $\lambda/80$ is not so very far removed from any physical significance as Mr Perry suggests, because an average good telescope objective would have, say, half the tolerance of spherical aberration. This represents an optical path difference of $\lambda/8$ between the centre and edge of the lens for axial pencils, or only one quarter of the distance from one visible interference band to the next. Naturally to measure this aberration I should then require an instrument which did not introduce errors of more than about one-tenth of the quantity I was proposing to measure, which brings us to $\lambda/80$ once more. For camera lenses $\lambda/10$ is amply sufficient; but I still maintain, from simple common sense considerations quite apart from any experimental results whatever, that to measure the aberrations of a good telescope objective, mirrors with an accuracy of $\lambda/80$ are very desirable.

I am glad Mr Smith has pointed out the dimensional inconsistency of the units employed for the various coefficients in this paper. As he says, the figures given are such as to yield path differences in wave-lengths or centimetres as the case may be, x and y being measured in cm. throughout.

As regards the tolerance for a_1, I have certainly made a mistake in writing it as ·00061. Excluding the possibility of correcting this by change of focus, the figure should be ·000018 as Mr Smith says. At the best focus, however, the tolerance becomes four times this amount, or ·00072. It seems to me that the best way of expressing all aberration coefficients is as so many multiples of the Rayleigh limit, referred to, say, the paraxial focus. Then if one lens had a coefficient which was ten times as great as the corresponding coefficient of another lens, we could say at once that the first lens was ten times as bad as the second, without the necessity of knowing either its aperture or focal length.

As regards Mr Smith's comment on the position of the stop plane in the two experiments, I may mention that I took great care that the cross-wires used in the interferometer experiments were placed in the identical position in which the Hartmann diaphragm was mounted for the previous experiments, namely, in contact with the back surface of the lens; and that in each case the assumption was made that the distance from the stop plane to the image was equal to the focal length. There is evidently some much more serious systematic error which has so far been entirely overlooked, and which accounts for the big difference between the interferometer and Hartmann results.

As regards Mr Smith's other points, the method of least squares is undoubtedly very unsatisfactory, but one is almost bound to use it for want of anything better. It has the added disadvantage of leading to rather laborious calculations, and I for

one am eagerly looking forward to the discovery of some much simpler and much sounder analysis of errors than has hitherto been available.

Mr Perry's suggestion of drawing accurate fine ink-lines on various surfaces of a lens is a very valuable one, as there is considerable difficulty in securing absolute squaring-on of the lens with cross-wires in only one plane; but it would of course be necessary to remove all the crosses except one, before attempting to view the pattern formed by an oblique pencil.

The errors stated in the paper obviously apply only to my one particular series of experiments, and I should no doubt get entirely different errors, were I to repeat them. But there seems little need to attempt to reduce the experimental errors below 1 per cent., as it appears to me that to obtain a real accuracy of this amount demands mirrors with irregularities of not more than $\lambda/80$.

I have no wish of course to suggest that the interferometer is useless as a means of measuring aberrations; on the contrary it appears to have a great future in that respect. The present paper gives the results of an attempt to analyse an interferogram—a rather specialised method of aberration measurement. If other users of any type of interferometer will describe their experiences in the measurement of aberrations, I shall be only too glad to hear of them.

THE EFFECT OF SLIGHTLY SELECTIVE ABSORPTION IN THE PAINT USED FOR PHOTOMETRIC INTEGRATORS

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MS. received, 12*th October*, 1926. *Read and discussed*, 11*th November*, 1926.

ABSTRACT. A sensibly non-selective internal coating for photometric integrators is very difficult to produce and still more difficult to maintain. It is clear that when lamps of different colour temperatures are compared in an integrator with an internal coating which shows selective absorption in, say, the blue, the values of candle-power obtained for the lamps of lower colour temperature will be too high and vice versa. The existence of this effect has long been known, but its magnitude has not hitherto been determined to any degree of accuracy. The paper describes a theoretical treatment of the problem and gives a simple method for calculating the magnitude of the effect in the case of sources having a spectrum approximating to that of a black body. The method has been tested by actual photometric measurement and a satisfactory agreement between theory and practice has been found. It is shown that in work on normal type electric lamps to an accuracy of 1 to 2 per cent. a quite noticeable coloration of the light may be produced by the sphere (either on account of paint or window selectivity or both) without the necessity for making any correction to the measured values of candle-power.

Introduction

The almost universal adoption of the mean spherical candle-power or lumen basis as the true measure of the output of modern light sources, particularly gas-filled electric incandescent lamps, has recently made it necessary to introduce an integrator, either sphere or cube, into all photometric test-rooms, and the special problems associated with the use of this form of apparatus have therefore taken on a much greater importance than heretofore. This is particularly true of the problem of finding a suitable paint for the interior coating of an integrator.

It may not be out of place here to describe briefly the method of making measurements of mean spherical candle-power or total flux. The lamp to be measured, and a standard of known mean spherical candle-power, are placed in turn inside a hollow sphere or cube as shown in Fig. 1. This sphere is whitened internally and is provided at one side with a small diffusing window *W*. The direct light from the lamp *L* is prevented from reaching the window by interposing a small white screen *S*. Thus the window receives only light which has been reflected from the walls of the sphere at least once and it can be shown* that, under the ideal conditions

* See, e.g., J. W. T. Walsh, *Photometry* (Constable, 1926) p. 206.

(*a*) that the lamp, screen, and window are all infinitely small compared with the size of the sphere, and (*b*) that the sphere coating is non-selective as regards colour and perfectly diffusing, then the light reaching the window is proportional to the mean spherical candle-power of the lamp. If this light be taken as proportional to the external brightness of the window, it is clear that the mean spherical candle-power of the unknown lamp can be found from that of the standard, by making measurements of the brightness of the window with each lamp in the sphere. Further it can be shown* that, if the average illumination of the sphere by direct light be E, and the reflection factor of the sphere coating be ρ, then the illumination of the window is $\rho E/(1-\rho)$ of which ρE is due to light which has suffered one reflection, $\rho^2 E$ is due to twice reflected light, and so on. It follows that, if $\rho = 0{\cdot}8$ (by no means a high figure), more than half the light reaching the window has suffered four or more reflections at the sphere surface.

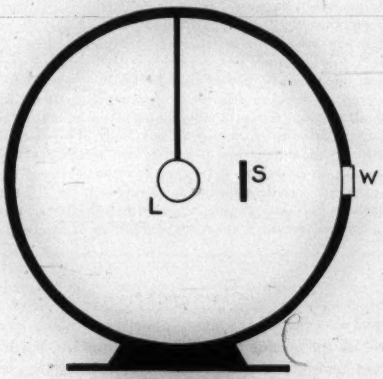


Fig. 1. The Integrating Sphere

The Effect of Selectivity

It will be clear from what has been said above that, if the sphere paint be not absolutely non-selective, the colouring effect, which might be quite unnoticeable by a single reflection, becomes so multiplied that it produces a very marked alteration in the colour of the light reaching the sphere window. In fact it is found in practice that none of the paints normally accepted as "white" will satisfactorily stand the test of repeated reflection within a photometric integrator. Further, a paint which shows a reasonably satisfactory approximation to whiteness when first applied soon becomes discoloured with lapse of time and within a few months the coating has to be renewed.

* J. W. T. Walsh, *loc. cit.*

It will be clear that the necessity for freedom from colour arises from the fact that it is seldom possible to have the test lamp and the standard with which it is to be compared giving light of exactly the same spectral distribution. Especially is this the case at the present time, for one of the most important applications of the integrating photometer is to the measurement of gas-filled lamps. The steadiness of candle-power of lamps of this type is not at present sufficient to make them suitable for use as sub-standards of candle-power. It follows, then, that vacuum lamps must be used as sub-standards, so that there is generally a considerable difference of colour between the test lamp and the sub-standard with which it is compared. Any selective absorption by the coating of the sphere therefore immediately introduces an error into the measurements, for if the coating be supposed selective in favour of the red, it is clear that the lamp giving the redder light, i.e. the vacuum lamp, will be unduly weighted, and the gas-filled test lamp will therefore have assigned to it too small a candle-power. The action of the sphere coating may be most clearly visualised by considering it equivalent to a coloured filter placed on the side of the photometer which faces the test lamp or the sub-standard.

Previous Work

After what has been said above it is not surprising to find that the subject of paint for photometric integrators has received a great deal of attention from all those who have worked with this type of apparatus. A very elaborate formula for a suitable paint was issued by the Verband Deutscher Elektrotechniker* and this is reproduced by R. Ulbricht in his classical work on the integrating sphere (*Das Kugelphotometer*; Oldenbourg, Munich, 1920). The Bureau of Standards has also given a formula for a satisfactory paint†. The effect of paint coloration has been treated by F. E. Cady‡ and by most of those who have dealt with the subject of integrating photometry in general§.

Quite recently R. C. Fox has described an ingenious method of compensating for the coloration of the paint‖. He neutralises the "reddish-filter" effect of the paint by inserting an actual bluish filter in front of the sphere window.

The matter has, naturally, not been neglected at the National Physical Laboratory. Great care has been expended in obtaining as white a covering as possible, and the apparently inevitable deterioration has been overcome by very frequent renewals. At the same time repeated attempts have been made to determine the magnitude of the effect in special cases. Two lamps operating at different efficiencies have been compared in the sphere just before repainting and the ratio of the mean spherical candle-powers thus found has been compared with the ratio obtained by direct measurement without a sphere. In no case was any definite and positive

* A. Utzinger, *E.T.Z.* **36** 1915 (137).

† A. H. Taylor, *Trans. Illum. Eng. Soc. N.Y.* **16** 1921 (587).

‡ F. E. Cady, *Journ. Franklin Inst.* **189** 1920 (787).

§ See, for instance, J. W. T. Walsh, *Photometry*, p. 217 and refs. (61) to (67); W. Dziobek, *Licht u. Lampe*, 1926 (395).

‖ R. C. Fox, *Proc. Opt. Convention*, **3** 1926 (289).

information obtained, the errors of measurement due to other causes masking the effect sought for in every case. The reason for this failure will appear later.

Theoretical Estimation of the Error

The negative result alluded to in the last section caused the authors to wonder whether the magnitude of the errors was really smaller than had previously been supposed. To test this it occurred to them that the effect might be treated theoretically by making certain simplifying assumptions which, to the degree of approximation necessary in calculating a quantity which was itself a correction, would probably be not too far from the truth.

It is found that, at any rate in the case of the paint used at the National Physical Laboratory, the effect of the gradual deterioration is to produce an effective progressive lowering of the colour temperature of the lamp in the sphere. For example, if a vacuum lamp operating at normal efficiency be matched in colour outside the sphere with a comparison lamp of similar type running at 100 volts, then it is found that, in order to obtain a colour match with the first lamp inside the sphere several months after the latter has been painted, the voltage on the comparison lamp has to be lowered to, say, 85 volts. It is, however, possible to obtain another colour match by making this voltage change, so that the effect of the sphere coating is in fact similar to that of one of the Wratten photometric filters. The simplifying assumption made is that the effect of the sphere coating is to change the spectral distribution of the light from the lamp in such a manner that it can be identified with the light given by a lamp of lower temperature, *in spectral distribution as well as in colour*.

If this assumption be granted it is at once possible to deduce from the amount of lowering of voltage of the comparison lamp the change of temperature produced, and therefore to find the spectral transmission curve of the colour filter to which the sphere coating is equivalent. This done, it becomes a simple matter to find the difference in the effect produced by this filter on the candle-powers of lamps of different efficiencies.

It should be noted that selective absorption by the diffusing window of the sphere produces exactly the same effect as selective absorption by the sphere paint. The voltage change on the comparison lamp is due to the combined effect and the result obtained below therefore applies to the combination.

The Calculation

If an electric lamp be run at such an efficiency that its colour temperature is T_1 the spectral distribution of the light emitted is given by Wien's law, viz.,

$$E_1 = C_1 \lambda^{-5} e^{-\frac{C_2}{\lambda T_1}},$$

If, after insertion in the sphere, the effective colour temperature be lowered to T_2, the spectral distribution is given by

$$E_2 = kC_1 \lambda^{-5} e^{-\frac{C_2}{\lambda T_2}},$$

where k is a constant independent of T or λ. The sphere thus acts as a colour filter for which the spectral transmission is given by

$$\tau_\lambda = k e^{-\frac{C_2}{\lambda T_1} + \frac{C_2}{\lambda T_2}},$$

or

$$\log_e \frac{\tau_\lambda}{k} = -\frac{C_2}{\lambda}\left(\frac{1}{T_2} - \frac{1}{T_1}\right).$$

The integral transmission factor of such a filter for light from a source of spectral distribution corresponding to temperature T can be found at once from the expression

$$\tau = \frac{\int \tau_\lambda K_\lambda E_\lambda d\lambda}{\int K_\lambda E_\lambda d\lambda},$$

where K_λ is the visibility function at wave-length λ.

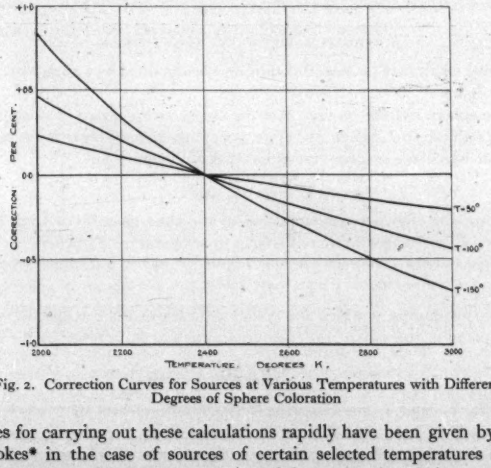


Fig. 2. Correction Curves for Sources at Various Temperatures with Different Degrees of Sphere Coloration

Tables for carrying out these calculations rapidly have been given by Buckley and Brookes* in the case of sources of certain selected temperatures and these tables have been used for the construction of the curves given below.

The method of obtaining the curves will be clear from the following example:

Efficiency of comparison lamp matching test lamp outside sphere = 9 lumens per watt.

Therefore colour temperature = 2400° K.

Efficiency of comparison lamp matching same lamp inside sphere = 6 lumens per watt.

Therefore colour temperature = 2250° K.

* H. Buckley and F. J. C. Brookes, *Illum. Eng.* **18** 1925 (241).

Hence
$$\log_e \frac{\tau_\lambda}{k} = -\frac{14330}{\lambda}\left(\frac{1}{2250} - \frac{1}{2400}\right)$$
$$= -\frac{0{\cdot}3981}{\lambda},$$

where λ is measured in microns.

Integral transmission for source at 2400° K = (50·024) k per cent.

Integral transmission for source at 2000° K = (50·444) k per cent.

Integral transmission for source at 3000° K = (49·678) k per cent.

Hence, if the sub-standard be a vacuum lamp at 9 lumens per watt (2400° K), the error introduced in measuring a lamp at 2·5 lumens per watt (2000° K) will be + 0·84 per cent. The curve marked "$T = 150°$" in Fig. 2 is obtained by calculating a number of values in this manner.

Curves for conditions of the sphere coating which correspond to smaller colour differences are also given in Fig. 2.

Alternative Method of Calculation

The same result may (at least theoretically) be obtained by a somewhat different method as follows:

Let the sphere coating be such that the colour temperature of a lamp at T_1 is apparently reduced to T_2. Then, as before, the sphere may be regarded as equivalent to a filter of which the spectral transmission curve is given by

$$\tau_\lambda = ke^{-\frac{C_2}{\lambda T_2} + \frac{C_2}{\lambda T_1}}.$$

It follows that the energy distribution of the light given by a lamp at colour temperature T_3 is changed to that of a lamp at temperature T_4, where

$$\frac{1}{T_2} - \frac{1}{T_1} = \frac{1}{T_4} - \frac{1}{T_3}.$$

If, now, two lamps of which the colour temperatures are respectively T_1 and T_3 be compared in the sphere, the measured ratio will be in error by the factor

$$\frac{I_2}{I_1} \times \frac{I_3}{I_4},$$

where I_1, I_2, I_3, and I_4 are respectively the candle-powers of a given lamp at temperatures T_1, T_2, T_3, and T_4*. If T_1, T_2, and T_3 be respectively 2400°, 2250°, and 2250° the relation shown above gives $T_4 = 2118°$. For a tungsten filament vacuum lamp the corresponding values of I_1, I_2, I_3, and I_4, found from the tables given by Middlekauff and Skogland† are as follows:

Temperature	*Relative Candle-power*
2400°	100
2250°	50·9
2118°	26·3$_2$.

* The assumption is made that the rate of change of the emissivity of tungsten throughout the visible spectrum is constant within the temperature range covered. This is approximately the case. See W. E. Forsythe and A. G. Worthing, *Com. Int. de l'Éclairage, Compte rendu*, 1924 (104).

† G. W. Middlekauff and J. F. Skogland, *Bull. Bur. Stand.* 11 1915 (483).

The value of $\frac{I_2 I_3}{I_1 I_4}$ is thus 0·986 which gives an error of 1·4 per cent. when comparing a lamp at 2250° K with one at 2400° K when the sphere coloration is 150° K. This value is about five times as large as that found by the first method, using Fig. 2, the difference being due to the large uncertainty in the values of candle-power found at the different temperatures. When it is remembered that a change of 10° in temperature involves a difference of about 3·8 per cent. in candle-power it will be seen that the first method of calculation is much to be preferred in practice.

Practical Verification of Theory

It will be seen from the result of the above calculations that the effect is very much less than had previously been supposed, and this accounts for the negative result obtained when efforts were made to measure it experimentally. An experimental verification has, however, been carried out since and it has shown that the order of magnitude found by theory is undoubtedly correct.

Table I

Expt.	K		Expt.	K	
	B.J.O.	C.J.M.		B.J.O.	C.J.M.
1	0·977	0·993	6	0·963	0·996
2	0·993	1·005	7	0·980	0·988
3	0·975	0·999	8	1·010	1·022
4	0·992	1·007	9	0·978	0·990
5	0·969	1·002	10	0·968	0·983
			11	0·983	0·979

Mean value of K = 0·989. Probable error of mean = 0·002.

There is maintained at the National Physical Laboratory a set of sub-standard lamps operating at an efficiency at which they match a pentane lamp, i.e. approximately 2 lumens per watt (colour temperature 1920° K). At the time of the experiments the sphere paint had been allowed to acquire such a colour that a normal efficiency vacuum lamp (2400° K) was lowered apparently about 110° K in colour temperature. Two carefully aged vacuum lamps were measured in a single direction at their normal efficiency and again at the efficiency at which they matched the pentane sub-standards. They were then measured inside the sphere at these same two efficiencies. If the candle-power of one of these lamps in terms of that of the other be represented by I, and if the values of I obtained as described above be as follows:

I_1 = candle-power in fixed direction at normal efficiency,
I_2 = candle-power in fixed direction at pentane efficiency,
I_3 = candle-power in sphere at normal efficiency,
I_4 = candle-power in sphere at pentane efficiency, then

$$\frac{I_1}{I_2} = K \frac{I_3}{I_4},$$

where K is the correction factor sought, and is equal to unity when there is no selectivity in the sphere.

The values of K obtained in twenty-two sets of measurements made by two observers working on eleven different occasions are shown in Table I.

Thus the value of the error determined by actual measurement is 1·1 per cent. The theoretical value calculated for this case by the first of the methods shown above is 0·7 per cent. so that the agreement between theory and practice is satisfactory.

Practical Application of the Results

It will be clear, from what has been said, that it is a comparatively simple matter to make the small correction necessary when comparing two lamps at different colour temperatures by means of a sphere the coating of which departs *slightly* from absolute whiteness. It is first necessary to determine the degree of coloration, and then to find the correction corresponding to this coloration for the particular temperature step involved in the comparison.

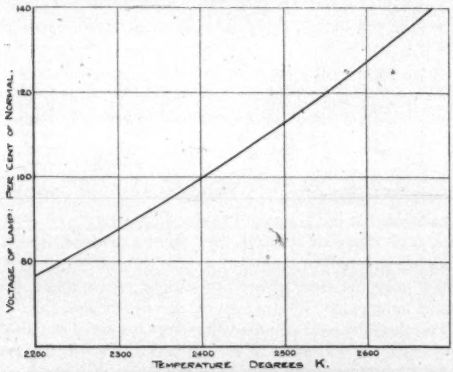


Fig. 3. Connection between Voltage and Temperature of a Tungsten Filament Vacuum Lamp

(*a*) *Determination of Coloration*

A lamp operating at about 2400° K (a tungsten filament vacuum lamp at an efficiency of 9 lumens per watt or 1·1 watts per mean horizontal candle) is colour-matched on the photometer bench with a comparison lamp of similar type. The first lamp is then placed inside the sphere and the voltage on the comparison lamp is altered until it colour-matches the light from the sphere window. The necessary voltage change, expressed as a percentage of the original voltage, will be denoted by V.

The temperature change corresponding with V is found from the curve of Fig. 3. This will be denoted by T and is the figure which expresses the coloration of the sphere coating.

(*b*) *Determination of the Correction to be applied to the Measurements of Candle-power*

The approximate colour temperature of the two lamps to be compared must first be found. In the case of a tungsten filament vacuum lamp the colour temperature may be found from the efficiency by means of the curve of Fig. 4*. In the case of a gas-filled lamp the connexion between colour temperature and efficiency is less definite, but an approximate figure may be obtained from Table II.

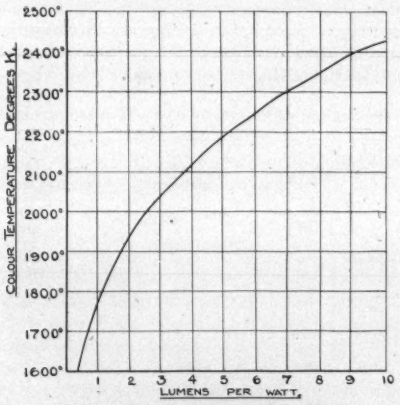


Fig. 4. Relation between Colour Temperature and Efficiency for a Tungsten Filament Vacuum Lamp

Let the colour temperatures of the two lamps to be compared be denoted by T_1 and T_2. Then on the diagram of Fig. 2 a curve must be imagined interpolated between the drawn curves, so that it corresponds to coloration temperature change T. The points on this interpolated curve corresponding to abscissae T_1 and T_2 are found, and the difference between the corresponding ordinates gives at once the correction required.

The efficiency scale in Fig. 4 is marked in lumens per watt. To a close approximation the efficiency in watts per mean horizontal candle is equal to the reciprocal of this figure multiplied by 9·85.

* H. Buckley, L. J. Collier, and F. J. C. Brookes, *Com. Int. de l'Éclairage, Compte rendu*, 1924 (219).

The following example will make the method clear:

(*a*) Voltage on comparison lamp to match lamp at 9 lumens per watt = 105·3.

Voltage on comparison lamp to match lamp when latter is in sphere = 93·7.

Therefore

$$V = \frac{105{\cdot}3 - 93{\cdot}7}{105{\cdot}3}$$

$$= 11{\cdot}0 \text{ per cent.}$$

Hence, from Fig. 3, $T = 90°$.

(*b*) Lamps to be compared are (i) tungsten filament vacuum lamp, 220 volts, 20 watts, (ii) gas-filled lamp, 110 volts, 200 watts, both at rated voltage.

The normal efficiencies for these lamps (taken from British Standard Specification 161) are

(i) 7·67 and (ii) 14·0 lumens per watt.

The corresponding colour temperatures are:

$T_1 = 2335°$ (see Fig. 4).

$T_2 = 2760°$ (see Table II).

Table II. *Approximate Efficiency and Colour Temperature of Normal Type Gas-filled Lamps*

Rating		Approximate efficiency*	Approximate colour temperature
	watts	lumens per watt	° K
100–130 volts	60	11·2	2690
	100	12·8	2730
	200	14·0	2760
	500	16·4	2850
	1000	17·5	2900
200–260 volts	60	9·3	2660
	100	11·2	2690
	200	12·8	2730
	500	14·5	2780
	1000	16·4	2850

On the diagram of Fig. 2 at abscissa 2335 the ordinate corresponding to an imaginary curve for $T = 90°$ is approximately + 0·05 per cent. The similar figure for 2760° is − 0·25 per cent. Hence the correction is + 0·3 per cent.

It is to be noticed that since the paint is reddish (lowers the colour temperature) the higher temperature lamp is penalised, and so the correction is to be applied as an *addition* to the measured candle-power supposing that the lower temperature lamp is taken as the sub-standard.

* The figures are approximately the means of the limits stated in British Standard Specification 161 (1924).

General Conclusions

It is clear, from the curves of Fig. 2, that in all normal cases of photometric work in the sphere or cube the correction factors can be kept down to less than 0·5 per cent. by repainting as soon as the colour correction has reached, say, 10 or 12 per cent. in voltage. Within these limits, therefore, it is probable that no correction need be made for ordinary commercial photometry*. For work to higher accuracy the correction to be applied in any particular case may readily be determined to within 0·1 per cent. by the method described above.

It will be clear that the investigation described in this paper deals only with the physical errors introduced into a photometric comparison by the selectivity of the paint or window. This selectivity has, of course, the further result of introducing a colour difference into photometric measurements, which would otherwise be made under homochromatic conditions. The errors inherent in a comparison involving a marked colour difference in the photometer head are psychological and it is clear that these can only be overcome by means of a colour filter† or by the use of a flicker photometer.

The authors wish to thank Messrs B. J. W. Oram and C. J. MacManus of the Photometry Division, National Physical Laboratory, for their careful work in making the photometric measurements summarised in Table I.

DISCUSSION

Mr B. P. Dudding: On behalf of my collaborators at our Research Laboratory at Wembley, I wish to congratulate the authors on their paper, and in particular, on the ingenious theoretical approximation by which the possible magnitude of the errors was estimated.

We have been investigating the same phenomenon from a rather different point of view and think our results and conclusions may be of some interest.

Late in 1919 it was decided to make all the photometric measurements in the G. E. C. Osram Lamp Factory in terms of mean spherical candles. For this purpose it was necessary to design an integrating photometer which would occupy the minimum floor space and yet be capable of measuring various types of lamps varying over a wide range of candle-power and operating temperature.

Our first design was arranged so that the travelling lamp (comparison lamp) moved vertically instead of horizontally, and the contrast cubes of the Lummer-Brodhun head arranged so that the window of the integrator was viewed directly. In order to multiply the range of this instrument a second diffusing window was used, whose position relative to the integrator window could be accurately adjusted by means of a micrometer screw.

* This conclusion is in agreement with the opinion expressed by F. E. Cady, *Journ. Franklin Inst.* **189** 1920 (787).

† See R. C. Fox, *loc. cit.*

This enabled us to measure, in a cube of 2 feet side, lamps whose candle-power was as small as four, without the illumination in the field of view being unduly low, and to extend measurements to 300 candles.

In using this instrument it soon became clear that errors were present which were outside those ordinarily associated with commercial photometry. Selectivity of the integrator was suspected as the cause.

We had not the advantage of sets of sub-standard lamps operating at widely different temperatures so had recourse to another method of determining the source of the error.

We determined the ratio of the mean spherical candle-power of a lamp measured in the integrator to its candle-power in a definite direction (measured on a standard photometer bench) over a range of temperature 2200° K to 2800° K.

The comparison lamp was operated at about 2500° K and kept constant throughout the measurements which were made independently by six observers.

We found that the reduction factor varied ± 3 per cent. from its value at colour match, i.e. this was the additional error that might be introduced when measuring a lamp which differed appreciably in operating temperature from the standard lamps used to calibrate the integrator.

At that time it was thought that this error was due to the paint used in the integrator and experiments were carried out using paints definitely tinted blue and yellow. This showed that the error was very little affected by visible changes in the paint and we could only assume it was due to selective properties of the diffusing windows used, possibly combined with the method of viewing them.

Further experiments were therefore made under the more usual conditions, i.e. with the window in the side of the integrator directly illuminating the screen of the Lummer-Brodhun photometer head. This showed that the change in reduction factor was now reduced to less than 1 per cent. (the probable experimental error), indicating clearly that the real cause of the larger errors was the introduction of the compound window or the method of viewing it. This normal arrangement was, however, not suited to our purpose, so experiments were made with the photometer head arranged to view directly the window in the side of the integrator. The result of these tests again showed the change of reduction factor to be less than 1 per cent. over the range of colour above mentioned, thus locating the main source of the error, in the first arrangement described, to the introduction of the second diffusing window.

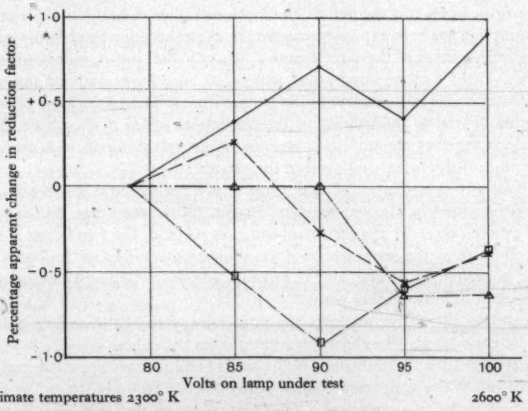
Further we carried out experiments to determine the error due to paint. A special photometric system was designed using Lummer-Brodhun prisms in such a way that a small portion of the inside surface of the integrator itself was viewed through a small hole in the side, this portion of the cube wall being screened from direct light. The comparison surface illuminated by the comparison lamp consisted of a screen of the same material treated with the same paint as the inside surface of the integrator. This method has the following advantages:

(1) A much closer approximation to a colour match can be obtained than is possible with the other arrangements.

(2) The brightness of the photometric field can be reduced by a sector disc placed between the aperture in the integrator and the Lummer-Brodhun prisms. The candle-power scale of the instrument can therefore be readily multiplied by a definite integer.

Repeating the experiments above described with this form of apparatus using the usual zinc white paint we found that the apparent change in the reduction factor of the lamp was of the order of $\frac{1}{2}$ per cent. over the range of temperature investigated.

When introducing definite colour tint to the paint, changes larger than 1 per cent were not recorded.



× Zinc white, Lummer-Brodhun photometer head illuminated by white opal window
□ Zinc white tinted blue, greenish opal window viewed directly
△ Zinc white, viewing internal surface or integrator
• Zinc white tinted blue, viewing internal surface of integrator

The results of these experiments of practical interest, covering the range of temperatures from approximately 2300 to 2600° K, have been plotted and are shown in the accompanying diagram where the percentage change in reduction factor is plotted as ordinate and the voltage on the test lamp as abscissa. The test lamp at 78 volts was at a colour match with the comparison lamp and the largest colour difference existed when the test lamp was operated at 100 volts.

During the progress of this work we have found greater consistency among the different observers when using this arrangement than when using any of those involving a diffusing window. This is of considerable importance commercially where one observer only is available for measurements.

Finally, we have found in our work using photo-electric cells that differences

in the colour of the paint may be much more important owing to the difference between the sensitivity curve of the cell and that of the eye.

Mr T. Smith: Messrs Walsh and Barnett have presented us with an interesting paper on a subject which, though well within the scope of this Society's work, has unfortunately been poorly represented in the past. Photometry is of the greatest importance to the general public, and an increase in the number of papers dealing with photometric problems would be generally welcomed.

The conclusions formulated in this paper are doubtless well supported by the authors' wide experience in the photometry of electric lamps, and emphasis might with advantage have been placed on this. The conclusions of the paper as it stands appear to rest upon two theoretical assumptions rather than upon experience; the first assumption has been expressly stated on p. 24, and is equivalent to the statement that k is independent of the wave-length λ, or, as I shall put it, independent of the frequency ω; the further assumption, which has been made without any explanation, is that k is also independent of the temperature t. The justification of the latter assumption is vital to the validity of the corrections given in the paper. The two assumptions together amount, as is obvious on general grounds, to a specification of the colour of the paint employed in the sphere.

Without any investigation it is obvious that if the spectral distribution of the light is determined by the temperature, any enclosed vessel may be used to give a mean candle-power by the direct substitution method, the two lamps inside the vessel being run at the same temperature. This method involves limitations from which it is desired to escape, and it becomes necessary to see what assumptions are implied in following other plans.

In all these alternative methods the properties of the paint and of the sphere have to be considered. The radiation issuing from the sphere is $\rho E/(1-\rho)$, where E is the average direct illumination of the sphere and is therefore a function of the frequency and of the temperature, and ρ, the reflection factor of the sphere coating, is a function of the frequency only. Observation shows that this represents radiation which matches in colour radiation from another source at another temperature. If the variation in the character of the radiation from both sources, due to changes of temperature is governed by the same law, the fact that a colour match is attained may be written

$$\frac{\rho}{1-\rho} f(\omega, T) \asymp k f(\omega, t),$$

where T is the temperature of the lamp inside the sphere and t that of the matching lamp outside. The sign $\asymp$ is used to denote that the equality is not physical but sensational. If, however, the match is physical this sign is to be replaced by the ordinary sign of equality, and it now becomes permissible to remove from f any factors involving ω only. If t only enters as an exponent $-\omega/t$, we have

$$k = \frac{\rho}{1-\rho} x^{\omega/t - \omega/T}.$$

Thus in general, since ρ is a function of ω, k depends on both the frequency and the temperature. If, however, the reflection coefficient of the paint satisfies the law

$$\rho = \frac{1}{1 + \theta\phi^{\omega}},$$

where θ and ϕ depend only on the age of the paint, we shall have

$$k\theta = \left(\frac{x^{1/t-1/T}}{\phi}\right)^{\omega},$$

so that the match will be physically perfect when

$$k = \frac{1}{\theta}, \quad \frac{1}{t} - \frac{1}{T} = \log_x \phi.$$

It appears to be much more important to select a paint which, both initially and as it ages, reflects according to this law than to select it for whiteness. For the practical problem mentioned in the paper a paint of a reddish hue is required; both white and blue are unsuitable.

If the conditions assumed in the paper are satisfied, i.e. if the paint is of this ideal kind, the properties of the eye do not enter into the problem, but the constancy of the difference between the reciprocals of the temperatures of the two lamps is fundamental. It seems therefore that the alternative method described on p. 26 is of greater significance than that given earlier. The fact that it indicates more uncertainty than the first method, suggests that the subject merits further attention. In passing it may be said that the use of a uniform scale of temperature reciprocals would be more convenient than a direct temperature scale for diagrams exhibiting the properties of electric lamps.

When the paint is not of the ideal type assumed in the paper but still such as to allow a colour match to be attained, the colour match equation is to be interpreted to mean

$$k_1 = k_2 = k_3 = k,$$

when the appropriate functions are inserted for K_1, K_2, K_3 in the equation

$$k_c = \frac{\int \frac{\rho}{1-\rho} K_c f(\omega, T)\, d\omega}{\int K_c f(\omega, t)\, d\omega}. \quad (c = 1, 2, 3.)$$

For a lamp run at a different efficiency we shall in general find three unequal quantities k_c' corresponding to temperatures T' and t'. If a satisfactory colour match can be made, it indicates that the differences $k_1' - k_2'$ and $k_2' - k_3'$ are small enough to be neglected, but it does not thereby follow that the values of each of the three k''s are approximately equal to the value found for the temperatures considered in the previous measurement, neither is it evident that the eye will select the match which makes

$$\frac{1}{t} - \frac{1}{T} = \frac{1}{t'} - \frac{1}{T'}.$$

If we assume that only the first power of the temperature changes in f need be considered, we shall have, on denoting the integrands in the numerator and denominator of the equation for k_e by N_e and D_e respectively, if we note that differentiation by the reciprocal of the temperature merely introduces a factor $b\omega$,

$$k_e' - k_e = b\left[\left(\frac{1}{T'} - \frac{1}{T}\right)\int N_e \omega\, d\omega - k_e'\left(\frac{1}{t'} - \frac{1}{t}\right)\int D_e \omega\, d\omega\right],$$

where b is a constant which does not vary from one source of the black body type to another. The second inverse temperature difference will be chosen to bring the differences between the k's within the limits to which the eye is insensitive. This means that in practice the spectral variation in the reflection coefficients of the paints used should be examined, for in the new integrals greater relative weight is attached to the light towards the blue end of the spectrum than in the old integrals.

Messrs Walsh and Barnett: We have been very interested in hearing about the work done by Mr Dudding and his colleagues and are gratified to find that their results confirm the conclusions we have reached in our work. It is, perhaps, worth noting that, in a recent communication, Dr Dziobek of the Reichsanstalt has mentioned the diffusing window as being the cause of large errors in measurements with the sphere.

Mr Smith has pointed out that we have made the assumption that k (see p. 24) is independent of temperature as well as frequency. It will be clear, however, that the proof of this independence is implicitly contained in the paper since it is assumed that a source of *any* temperature (within the range considered), if placed in the sphere, may be matched in colour by a source at some appropriate different temperature. If, then, the relation $T_\lambda = ke^{(c_2/\lambda)(1/T_1 - 1/T_2)}$ holds for a given pair of temperatures T_1 and T_2, an exactly similar relation holds for any other pair of temperatures, T_3 and T_4 say, provided the value of T_4 be properly chosen with respect to T_3. Thus

$$T_\lambda = k' e^{(c_2/\lambda)(1/T_3 - 1/T_4)}$$

and therefore

$$k x^{(1/T_1 - 1/T_2)} = k' x^{(1/T_3 - 1/T_4)} = \ldots,$$

where x is written for $e^{c_2/\lambda}$.

This equation gives

$$\log(k/k') = (c_2/\lambda)\,[(1/T_3 - 1/T_4) - (1/T_1 - 1/T_2)].$$

Since k and k' are independent of λ it follows that the value of the expression in square brackets must be 0 or ∞. Hence $k = k'$, i.e. k is independent of temperature as well as frequency.

The treatment of the problem for the case of identity of colour sensation without identity of spectral distribution is very interesting. For practical purposes, however, we doubt if there would be much value in obtaining spectrophotometric data for the sphere coloration. A colour difference corresponding to 100° at 2400° only involves a change of about 12 per cent. in the ratio of the intensities at 450 and 700$m\mu$ so that "second order" departures from the assumption made on p. 24 would be difficult to measure with any degree of certainty.

AN ACCURATE METHOD OF ASCERTAINING THE POSITION OF THE FOCAL POINT OF AN OPTICAL SYSTEM

By CONRAD BECK, C.B.E.

MS. received, 27th October, 1926. Read and discussed, 11th November, 1926.

ABSTRACT. A simple and accurate method of finding the positions of the foci of different zones of microscope object glasses is described. It consists essentially in placing a diaphragm with two slit apertures behind the object glass to be tested, the directions of the slits being at right angles to one another, and finding the position where the images form a symmetrical cross. Results of measurements by this method of the zonal aberrations of apochromatic microscope object glasses are given.

Introduction

In testing the zonal aberrations of a microscope object glass I have tried many ways of ascertaining the position of the best focal point for each zone, and having devised a plan that gives better results than all those I have yet tried, I describe the method in the hope that it may be of use in dealing with physical measurements.

Many indirect methods have been employed, such as the Hartmann test, the Hartridge test, the comparison of the diffraction rings on both sides of the focus, and the alteration of the conjugate images on the star test. In my hands the last is the only method that has given consistent results over a large number of repeated tests, but the disadvantage of it is that it is extremely difficult to interpret the results except in the case where the aberrations are practically eliminated. It will indicate when there are no perceptible aberrations with an accuracy that perhaps is not yet exceeded, but, when aberrations are present, it is extremely difficult to interpret their nature. It may be that the method of testing by the interferometer, when worked out, will have greater accuracy than the method here described, but the complexity of the interferometer method involves the possibility of considerable errors unless the greatest precautions are taken, and the plan here described may have advantages for practical work due to its simplicity.

It is well known that even when considerable aberrations are present a direct observation of an image will not enable the positions of the focal points of the zones of a microscope object glass to be determined with any accuracy. This is not surprising because the image of a point under the high magnification of a microscope is a diffraction disc and a section of the cone of light forming the image is of the shape shown in Fig. 1, in which a definite portion of the bundle of light on either side of the focus is contained in a parallel cylindrical envelope and has no definite focus. There is an actual depth of focus which depends on the angle

of the cone of light forming the image and, provided the different foci lie within this region, their position cannot be determined. The actual distance over which this difference in focus cannot be observed is greater than the length of the above mentioned cylindrical envelope due to the accommodation of the eye and many other causes.

In devising the test which has proved to be successful, the object was to produce an image which assumed a particular shape when it was in focus, so that, instead of estimating by the eye when the image was sharp, one could observe when it was of a particular shape.

Description of Method

After many experiments, the following method was adopted. To determine the focal point of a zone of a lens, which may be represented by the white ring in Fig. 2, a diaphragm with two rectangular apertures on opposite sides of the centre

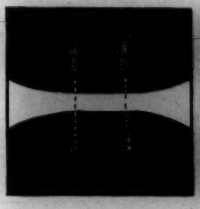


Fig. 1

Fig. 2

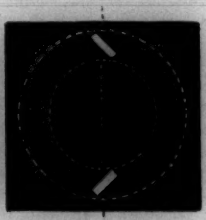


Fig. 3

is placed behind the lens, so that the apertures are situated within the zone to be tested (Fig. 3). Each of these apertures is a small slit about three times as long as

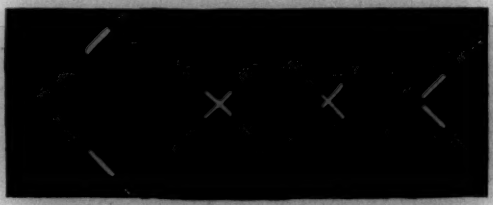


Fig. 4 Fig. 5 Fig. 6 Fig. 7

it is broad. They are placed accurately across the diameter of the disc and at right-angles to each other. If a fine point source of light is used as an object, each slit forms a diffraction pattern in the image plane (Fig. 4), the central portion of which

is a band at right angles to the length of the slit apertures. At the position where the two images meet at a focus the appearance is that of a perfectly symmetrical cross, as shown in Fig. 5. Where they are out of focus, the appearance is as shown in Figs. 6 and 7. Thus the crossing point of the two bundles of light is ascertained by focussing to the position where the figure is symmetrical, and I find in practice that this position can be set with great precision.

Various shapes of diaphragms have been tried, many of which produce very elaborate and beautiful patterns, but so far a symmetrical cross pattern has proved to result in the most accurate setting.

The method of procedure is as follows. A silvered surface on glass affords a number of pinholes of varying sizes and one can be selected which is sufficiently small to produce a satisfactory diffraction pattern when used as an object. It must be illuminated with a powerful light by means of a wide angle condenser. Such an illuminated pinhole is focussed for a particular tube length of the microscope, after which the object glass must not be shifted during the whole period of the tests, and greater rigidity than is possessed by most microscopes is essential.

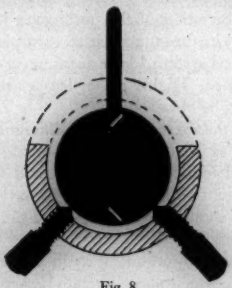


Fig. 8

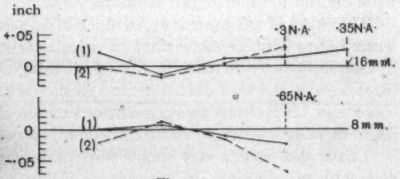


Fig. 9

A series of diaphragms must be prepared with rectangular apertures of the nature described, which are at different distances from the centre of the disc, so that all zones of the lens can be tested by changing the diaphragms. These must be changed without any risk of moving the object glass, and I find in practice that, if they drop into a slot which they loosely fit and rest on two adjustable points as in Fig. 8, they can be interchanged without fear of displacing the object glass. The microscope is placed with its body approximately horizontal and an adapter with a slot and adjustable points screwed between the object glass and the body. The focussing must now be done by moving the eyepiece alone and this should be supported on a graduated slide of a very good quality which is not connected directly to the body of the microscope. The ordinary rack and pinion draw-tube of the microscope can be used, but will not give very accurate results. I use a scale which reads to thousandths of an inch, and, by taking the mean of six readings for each observation, have obtained good results.

Results

The diagrams in Fig. 9 represent the focal points of four zones of two 16 mm. and two 8 mm. apochromatic object glasses of different makes set for the tube lengths for which they are best corrected. The 16 mm. lens No. (1) has a larger aperture than No. (2). The actual distances of the zone focus from the best average focus are as follows:

16 mm. Zone (inch)	Lens (1) (inch)	Lens (2) (inch)	8 mm. Zone (inch)	Lens (1) (inch)	Lens (2) (inch)
·12	+·022	−·002	·12	·000	−·020
·2	−·014	−·016	·2	+·009	+·011
·3	−·010	+·004	·3	−·010	−·015
·39	+·012	+·030	·4	−·023	−·066
·43	+·024				

Judging from the variations in a large number of readings, I should estimate that these figures are probably correct to about ± ·005 inch, although the most central zone is difficult to set and is less accurate. When the tube length of one of these lenses is increased by 30 mm., the aberration of the outer zone is more than three times as great. The 16 mm. lens (1) has been taken as an example as it is the lens with the least aberration of any optical system I have seen; nevertheless, errors of a noticeable amount can be measured. Higher power lenses are more difficult to calibrate but satisfactory results can be obtained.

The object of this paper is not so much to discuss the aberrations of the different lenses but to describe the method of measurement as it would appear that small measurements may be made by this means using a microscope or telescope and a diaphragm of the kind described, and in such a case the diaphragm might have longer slits. It probably would not be as accurate as an interferometer method but might be more convenient and considerably less costly.

I have also made a very simple form of diaphragm by means of a slit and a plate with two diagonal slit apertures as in Fig. 10, which slides laterally across the slit, thus leaving a pair of apertures which may be made to travel outwards over the different zones of the lens, but the shape of the apertures so produced does not give good diffraction patterns and so far the fixed rectangular apertures are more satisfactory.

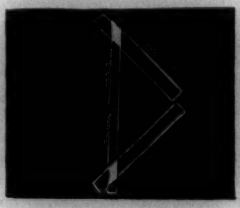


Fig. 10

DISCUSSION

Dr John S. Anderson: Anyone who has tried to obtain the exact position where the diffraction images from two circular apertures coincide will at once appreciate the fact that Mr Beck's simple and ingenious method of obtaining a symmetrical pattern marks a distinct improvement in technique. I imagine that the principle will be of considerable use in many tests.

Mr R. Kingslake: I should be very glad if Mr Beck could give any idea as to the method of cutting the slots in the diaphragm, when that is sufficiently small to use with a $\frac{1}{12}$ inch objective. This method of locating foci is very interesting and should be conveniently applicable to telescope objectives, using a real or artificial star. The chief difficulty in the practical application of the method would appear to be, not so much in the visual estimation of the position of the exactly symmetrical cross, but rather in measuring the longitudinal distance between the foci thus found. Undoubtedly the method has the advantage that the aberration is measured directly; but it has the accompanying disadvantage that the quantity to be measured is generally very small. In the Hartmann test, for example, the measurements on the two photographs are large, but the final result is obtained by calculation from two measured quantities, each of which may have experimental errors.

Prof. Alan Pollard: Mr Beck very kindly demonstrated his method to me some time ago and I suggested to him the trial of small annular openings in place of the rectangular slots. Such openings should not be difficult to construct. The want of symmetry in the diffraction pattern due to these openings when their images do not exactly coincide might be more striking than that due to the rectangular slots and I would like to ask Mr Beck if he has tried this suggestion. But I would now like to make the further suggestion that these two annular openings have different diameters so that their images will be concentric instead of overlapping when their centres coincide.

I would like to add that Mr Beck also demonstrated to me an experiment which he has not described and which indicates in a striking manner the sensitiveness of the method. If a red filter is held before the eye and the ocular moved until the images of the slots cross symmetrically, the symmetry is at once disturbed by substituting a blue for the red filter, though a well-corrected objective be used.

Mr T. Smith: Can Mr Beck say how the accuracy attainable in these measurements compares with the latitude in longitudinal displacements of the focus due to the undulatory character of light?

Mr Beck: In reply to the question of Mr Kingslake, it is essential that the two slots used for any one experiment should be similar in dimensions and correctly placed to a considerable degree of accuracy. This was accomplished by means of a very accurately divided pair of slides used as the supporting table of a special drilling machine made for the purpose of producing drilled holes in metal in predetermined positions.

I have tried the method suggested by Prof. Pollard of using annular openings instead of rectangular slots, but the combination of two annular diffraction patterns arriving at a focus at different angles did not produce a diffraction pattern which was sufficiently characteristic to be of assistance. I had hoped that there might be a shape which would produce a combined diffraction pattern more sensitive than a cross, but the small size of the apertures required for testing microscope lenses tends to limit one to simple shapes. It must be remembered that the diffraction patterns are not dioptric and have a tendency to produce a combined pattern not necessarily corresponding to the shape of the apertures.

In reply to Mr Smith, the test is not quite as accurate a means of detecting the presence of zonal errors as the conjugate star test, but it gives actual quantitative results with no errors beyond the actual errors of setting, while the other method is difficult to translate. The errors in the different zones of a microscope object glass that are measurable are very large compared with the wave-length of light as regards the position of the image, but, if converted to the error in the conjugate position of the object, are of the order of a small fraction of a wave-length. My general experience leads me to believe that, if a comparatively small zone of a microscope object glass has a lateral aberration in the image not more than $2\frac{1}{2}$ or 3 times the diameter of the diffraction disc formed by a theoretically perfect system, the lack of definition so produced cannot be detected, and some of the best modern apochromatic microscope object glasses fulfil this condition.

REVIEW

Field Astronomy for Engineers and Surveyors. By DAVID CLARK, M.A., B.Sc., M.Inst.C.E. Pp. viii + 164; 50 Figs. (London: Constable and Co., Ltd., 1926.) Price 10*s.* 6*d.* net.

Prof. Clark's little treatise forms an excellent textbook in such astronomy as is needed for the surveyor in determining latitude, longitude and the direction of the meridian. The book is divided into three parts: "Principles," including the formulae for solving the celestial triangle; "Instruments," which includes descriptions of most of the theodolites, etc. used in the field; and "Observations," in which section are discussed the various methods to be adopted in determining time, azimuth, latitude, and longitude. Each of these sections is admirably written but to one who has been accustomed to dealing with the similar problems that arise in navigation it is rather a matter for surprise that among the methods of solving the spherical triangle there should be no reference to the haversine formulae that are now so extensively used at sea. Possibly the explanation may be that in Inman's *Nautical Tables* (where log haversines and natural haversines are tabulated side by side) they are given to five figures only. This is good enough for navigation but not good enough for the surveyor. If that is the case it might be worth consideration to get such tables published in a more extended form since the routine work of computing positions from observations is certainly simplified by the use of these formulae.

Alternatively the development of formulae and tables specially adapted to solving a spherical triangle with the help of a calculating machine has a good deal to be said for it.

Another criticism that can be made against Prof. Clark's book is the meagre space allotted to the prismatic astrolabe, an instrument which is being more and more used for accurate and rapid field work. The astrolabe has much to recommend it. The determination of position upon the earth's surface is made from a measurement of the direction of the vertical at the spot relative to the stars. With theodolites, transit instruments, and zenith telescopes the mechanical axis has to be set up vertically by means of levels and the precautions that are necessary to avoid upsetting this verticality are very considerable. With the prismatic astrolabe the vertical is looked after by gravity itself and only the roughest levelling of the instrument is necessary. Further in the astrolabe there are no circular scales to be read. The measurement is made always with reference to the fixed angle between the two faces of a prism, and, used intelligently, this angle can be relied on to remain constant to a very small fraction of a second.

Whether such refinement of accuracy has any meaning is a very doubtful point. Prof. Clark speaks on p. 116 of a determination of latitude by Talcott's method with a probable error not exceeding $\pm 0^{s}\cdot 10$. The smallness of that probable error depends, among other things, upon the optical axis of the telescope remaining at a constant inclination to the vertical. The tenth of a second will go by the board of the objective if the telescope shifts laterally in its cell by a couple of wave-lengths of sodium light as the telescope is swung in azimuth from north to south. Add to this the possibility of a minute change in the direction of the vertical axis as the observer shifts his position from one side to the other; include the possibility of a slight warping of the telescope tube when the instrument is turned to expose first one side and then the other to a cold wind; include the many other sources of error that will not cancel out and, in the end, there remains, at all events in the mind of the reviewer, a considerable mistrust of that tenth of a second probable error. It would be interesting to see the result if, say, fifty different observers determined the latitude of the same spot on different nights extending over a fairly long period and then plotted the position of the spot on the same map. If the probable error of $\pm 0^{s}\cdot 10$ could be attained half of the plotted points should lie within a space of 20 feet. Would they?

T. Y. B.

ABSTRACTS OF PATENT SPECIFICATIONS

252779. Brown, W. J., and others. STROBOSCOPIC APPARATUS.

Intermittent illumination is provided by a quick-quenching lamp, which is controlled by an oscillating circuit including a triode valve. Operative connexion with the moving part to be examined is thus unnecessary.

252880. Benjamin Electric Ltd., and another. REFLECTORS.

A reflector, generally parabolic in shape, is built up with a number of plane elements, each approximately equal in area to the source of light. When silvered glass is employed, the silver is preferably backed with lead.

253639. Wilson, J. S., and another. MEASURING ANGLES.

The images of two opposite points upon a divided circle are simultaneously thrown into the eyepiece, which is fitted with an arrangement including two movable indices in conjunction with a micrometer to give a direct reading of the smaller units in the mean result.

253957. Baird, J. L. TELEVISION APPARATUS.

In front of the object is rotated a carrier bearing a number of lenses suitably arranged to search the field. To obtain a finer subdivision of the image, the shaft upon which the carrier is mounted may be simultaneously rocked at right-angles to its length, or a shutter, reflector, or other optical means may be employed for this purpose. The varying beam impinges upon photo-sensitive cells, the resulting electrical impulses being transmitted to the receiving station, where a synchronised apparatus is located.

254180. Saunders, J. S. REFLECTORS.

In a reflector for vehicle lamps, a substantially paraboloidal upper part merges into parabolic sides and a substantially plane base. The source of light may be placed unsymmetrically to throw more light to one side than to the other.

254181. August, J. R. C., and another. OPTICAL APPARATUS.

The movement of object, lens, and screen is controlled throughout the range of the instrument by a drum or plate formed with cam guides to engage pins upon the parts carrying the optical elements. The shape of the guides may be such that the linear size of the image increases uniformly with the rotation of the drum.

254290*. Zeiss, C. PRISMS.

A reversing prism consisting of two parts cemented together provides five reflections, of which one takes place at a roof surface. The light emerges at a face at which reflection has previously taken place, in such a way as to afford direct parallel vision.

254301*. Brock and Weymouth Inc. SURVEYING.

In converting the conical projection of a series of photographs into the orthographic projection of a map, a series of contoured plates of the view, and a geometrical traverse of a number of points, including the optical centres of each photograph, are prepared. Each plate, corrected for scale and position, is then projected on the traverse drawing, and the contours successively drawn, adjustment as from line to line being effected automatically.

R. J. T.

* Specification published before acceptance.